Section 6.1

Net Change

Recall for a differentiable function (at a) $y = f(x)$, the derivative of f at a gives a linear approximation $L(x) = f'(a)(x-a)+f(a)$. Then we can make an approximation of the change in f over a small interval around a by

$$\begin{aligned}\Delta y = f(x_1) - f(x_2) &\approx L(x_1) - L(x_2)\\ &= \big(f'(a)(x_1 - a) + f(a)\big) - \big(f'(a)(x_2 - a) + f(a)\big)\\ &= f'(a)(x_1 - x_2)\\ &= f'(a)\Delta x\end{aligned}$$

For infinitely small changes in x we write $dy = f'(a)dx$.

A total or net change over a larger interval can then be approximated by

$$\text{Net Change} \approx \sum_{i=1}^{n} f'(x_i^*)\Delta x.$$

by taking the appropriate limit we see

$$\text{Net Change} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f'(x_i^*)\Delta x =$$

In general, if $A(x)$ is a function that gives some total measurement for a given x, and if the measurement A can be approximated by a linear relation for small segments or pieces by $A \approx f(c)\Delta x$ for some function $f(x)$, then the net change in $A(x)$ over the interval $[a, b]$ is given by

$$A(b) - A(a) = \int_a^b f(t)dt$$

Example 6.1.1. If an object moves at a constant velocity of $v = 4$ m/s for 5 seconds, how far has the object moved?

If the objects velocity changes over time, say $v(t) = t^2$, how far has the object moved in 5 seconds?

If the function $A(x)$ is unknown, it can be recovered from the net change formula by

$$A(x) = A(0) + \int_a^x f(t)dt$$

Example 6.1.2. Suppose a culture of cells has a population of 100 cell when nutrients are added at time $t = 0$. If the total population $N(t)$ increases at a rate of $N'(t) = 90e^{-0.1t}$, find the population at time $t = 10$.

Find a formula for $N(t)$.

Example 6.1.3. A winch located atop a 30 m building is pulling a steal cable from the ground weighing 4 N/m. If the cable reaches from the winch to the ground, what is the total amount of work that is done by the wench in pulling the cable completely up?

Example 6.1.4. Suppose a thin 1 meter rod has linear density (measured in kg/m) down its length given by the linear density function $\rho(x) = x\sqrt{2-x^2}$ where x is measured in meters. Find the rod's total mass.

Example 6.1.5. Hooke's Law states the force F required to keep a spring in a compressed or stretched position x units from equilibrium is directly proportional to x. Suppose it takes a force of 30 N to stretch and hold a spring .2 m for its equilibrium position. How much work is required to stretch the spring 0.3 m from its equilibrium?

Example 6.1.6. A leaky bucket weighing 1 kg holds 3 kg of water. How much work is required to raise the bucket vertically a distance of 10 m if the water leaks from the bucket at a constant rate of 0.2 kg/m?

Example 6.1.7. A rectangular dam with a width of 40 m and depth of 10 m spans the head waters of a lake. If the water level is at the top of the dam, what force is exerted on the dam by the water?

Example 6.1.8. A plane flying horizontally in a straight line at an altitude of 1 mile passes overhead of an observer. If the plane passes the observer at time $t = 0$ traveling 100 knots and has a constant acceleration of 4 knots/sec, what is the rate at which the viewing angle is changing at time $t = 20$ sec?